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THE BEST CONSTANT IN AN INEQUALITY OF OSTROWSKI TYPE

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ABSTRACT. We prove that the constant $\frac{1}{2}$ in Dragomir-Wang's inequality [2] is best.

1 INTRODUCTION

The classical inequality of Ostrowski, [1, p. 469] is

Theorem 1.1. *Let I be an interval in \mathbf{R} , I° the interior of I , $f : I \rightarrow \mathbf{R}$ be differentiable on I° . Let $a, b \in I^\circ$ with $a < b$ and $\|f'\|_\infty = \sup_{t \in [a, b]} |f'(t)| < \infty$.*

Then

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ in (1.1) is the best possible.

For, suppose that

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[k + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in [a, b]$. Taking $f(x) = x$, gives $\|f'\|_\infty = 1$ and (1.2) becomes

$$\left| x - \frac{a+b}{2} \right| \leq \left[k + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)$$

for all $x \in [a, b]$. With $x = a$ this becomes

$$\frac{b-a}{2} \leq \left(k + \frac{1}{4} \right) (b-a)$$

giving $k \geq \frac{1}{4}$.

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2 THE RESULTS

In [2], Dragomir and Wang obtained a related inequality:

Theorem 2.1. *Let I, f, a, b be as above and $f' \in L_1[a, b]$. Then*

$$(2.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{2} + \frac{|x - \frac{a+b}{2}|^2}{b-a} \right] \|f'\|_1$$

for all $x \in [a, b]$,

but did not prove that the constant $\frac{1}{2}$ is the best possible one.

In [3], S.S. Dragomir gave an extension of Theorem 2.1 for mappings with bounded variation, i.e., he proved the result:

Theorem 2.2. *Let $f : [a, b] \rightarrow \mathbf{R}$ be a mapping with bounded variation on $[a, b]$. Then for all $x \in [a, b]$, we have the inequality:*

$$(2.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{2} + \frac{|x - \frac{a+b}{2}|^2}{b-a} \right] \bigvee_a^b(f)$$

where $\bigvee_a^b(f)$ denotes the total variation of f on $[a, b]$.

The constant $\frac{1}{2}$ is the best possible one.

For the sake of completeness and as the paper [3] is not published yet, we give here a short proof of Theorem 2.2.

Using the integration by parts formula for Riemann-Stieltjes integral, we have

$$(2.3) \quad \int_a^b p(x, t) df(t) = f(x)(b-a) - \int_a^b f(t) dt$$

where

$$p(x, t) := \begin{cases} t-a & \text{if } t \in [a, x) \\ t-b & \text{if } t \in [x, b]. \end{cases}$$

for all $x, t \in [a, b]$.

It is well known that if $p : [a, b] \rightarrow \mathbf{R}$ is continuous on $[a, b]$ and $v : [a, b] \rightarrow \mathbf{R}$ is with bounded variation on $[a, b]$, then

$$(2.4) \quad \left| \int_a^b p(x) dv(x) \right| \leq \sup_{x \in [a, b]} |p(x)| \bigvee_a^b(v).$$

Applying the inequality (2.4) for $p(x, \cdot)$ and f , we get

$$\left| \int_a^b p(x, t) df(t) \right| \leq \sup_{t \in [a, b]} |p(x, t)| \bigvee_a^b(f)$$

$$= \max \{x - a, b - x\} \bigvee_a^b(f) = \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f).$$

Using the identity (2.3), we deduce the desired result (2.2).

To prove the sharpness of the constant $\frac{1}{2}$ in the class of mappings with bounded variation, assume that the inequality (2.2) holds with a constant $C > 0$, i.e.,

$$(2.5) \quad \left| \int_a^b f(t) dt - f(x)(b-a) \right| \leq \left[C(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f),$$

for all $x \in [a, b]$.

Consider the mapping $f : [a, b] \rightarrow \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \in [a, b] \setminus \left\{ \frac{a+b}{2} \right\} \\ 1 & \text{if } x = \frac{a+b}{2} \end{cases}$$

in (2.5). Then f is with bounded variation on $[a, b]$ and

$$\bigvee_a^b(f) = 2, \quad \int_a^b f(t) dt = 0$$

and for $x = \frac{a+b}{2}$ we get in (2.5), $1 \leq 2C$; which implies that $C \geq \frac{1}{2}$ and the theorem is completely proved. ■

Now, it is clear that if f is differentiable on (a, b) and $f' \in L_1[a, b]$, then f is with bounded variation on $[a, b]$ and applying Theorem 2.2 we get Theorem 2.1. But we are not sure that the constant $\frac{1}{2}$ is best in the class of differentiable mappings whose derivatives are in $L_1(a, b)$. We give an example showing that the constant $\frac{1}{2}$ remains best for this class of mappings, too.

Suppose that

$$(2.6) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[k + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right] \|f'\|_1, \quad x \in [a, b].$$

Let C be any positive real and let

$$f(x) = \frac{C}{C^2 + x^2} - \tan^{-1} \left(\frac{1}{C} \right)$$

with $a = -1$ and $b = 1$.

Direct calculation shows that $\int_a^b f(t) dt = 0$.

Also, since $f'(x) \leq 0$ for all $x \geq 0$,

$$\|f'\|_1 = 2 \int_0^1 |f'(t)| dt = -2 \int_0^1 f'(t) dt = 2[f(0) - f(1)]$$

$$= 2 \left[\frac{1}{C} - \frac{C}{C^2 + 1} \right] = \frac{2}{C(C^2 + 1)}.$$

Substituting these into (2.6) and taking $x = 0$ then gives

$$\left| \frac{1}{C} - \tan^{-1} \left(\frac{1}{C} \right) \right| \leq k \frac{2}{C(C^2 + 1)}$$

so that

$$k \geq \frac{C^2 + 1}{2} \left[1 - C \tan^{-1} \left(\frac{1}{C} \right) \right].$$

Since the right side tends to $\frac{1}{2}$ as $C \rightarrow 0+$, we get $k \geq \frac{1}{2}$, which shows that the constant $\frac{1}{2}$ is the best possible in Theorem 2.1. ■

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